

A Mathematical Proof of Dodgson's Algorithm

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ABSTRACT. In this paper we give a mathematical proof of Dodgson algorithm [1]. Recently Zeilberger [2] gave a bijective proof. Our techniques are based on determinant properties and they are obtained by induction.

1. Introduction

To prove mathematically the well-known Dodgson's algorithm, concerning a square matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ we may generalize it as follows:

$$(1.1) \quad \det \left[(a_{i,j})_{1 \leq i,j \leq n} \right] \det \left[(a_{i,j})_{\substack{i \neq k,l \\ j \neq k,l}} \right] = \det \left[(a_{i,j})_{\substack{i \neq l \\ j \neq l}} \right] \det \left[(a_{i,j})_{\substack{i \neq k \\ j \neq k}} \right] - \det \left[(a_{i,j})_{\substack{i \neq l \\ j \neq k}} \right] \det \left[(a_{i,j})_{\substack{i \neq k \\ j \neq l}} \right],$$

for all $k, l = 1, \dots, n$ considering $k < l$.

For this purpose, we need some notations

1.1. Notations.

NOTATION 1. The $(n-k) \times (n-l)$ matrix obtained from A by removing the $i_1^{th}, i_2^{th}, \dots, i_k^{th}$ rows and the $j_1^{th}, j_2^{th}, \dots, j_l^{th}$ columns is denoted by $(a_{i,j})_{\substack{i \neq i_1, i_2, \dots, i_k \\ j \neq j_1, j_2, \dots, j_l}}$.

NOTATION 2. we denote by $A(k)$ to the $(n-2)$ square matrix obtained from A by removing the rows $(n-2)$ and $(n-1)$ and the columns $(n-1)$ and n and replacing the column $(n-2)$ by the $(n-k)$, i.e

$$(1.2) \quad A(k) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-3)} & a_{1(n-k)} \\ a_{21} & a_{22} & \cdots & a_{2(n-3)} & a_{2(n-k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-3)1} & a_{(n-3)2} & \cdots & a_{(n-3)(n-3)} & a_{(n-3)(n-k)} \\ a_{n1} & a_{n2} & \cdots & a_{n(n-3)} & a_{n(n-k)} \end{pmatrix},$$

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also it can be written as a block matrix :

$$A(k) = \begin{pmatrix} & a_{1(n-k)} \\ & a_{2(n-k)} \\ (a_{i,j})_{\substack{i \neq n, n-1, n-2 \\ j \neq n-1, n-2}} & \vdots \\ & a_{(n-3)(n-k)} \\ & a_{n(n-k)} \end{pmatrix}.$$

NOTATION 3. The n square matrix obtained from A by replacing the row n by the $(n-l)$ one is denoted by $B(l)$, i.e

$$(1.3) \quad B(l) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-3)} & a_{1(n-2)} & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(n-3)} & a_{2(n-2)} & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{(n-3)1} & a_{(n-3)2} & \cdots & a_{(n-3)(n-3)} & a_{(n-3)(n-2)} & a_{(n-3)(n-1)} & a_{(n-3)n} \\ a_{(n-2)1} & a_{(n-2)2} & \cdots & a_{(n-2)(n-3)} & a_{(n-2)(n-2)} & a_{(n-2)(n-1)} & a_{(n-2)n} \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)(n-3)} & a_{(n-1)(n-2)} & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{(n-l)1} & a_{(n-l)2} & \cdots & a_{(n-l)(n-3)} & a_{(n-l)(n-2)} & a_{(n-l)(n-1)} & a_{(n-l)n} \end{pmatrix}.$$

2. RESULTS

We need some lemmas:

LEMMA 1. We have

$$(2.1) \quad A(0) = (a_{i,j})_{\substack{i \neq n-1, n-2 \\ j \neq n-1, n-2}}, A(1) = (a_{i,j})_{\substack{i \neq n-1, n-2 \\ j \neq n, n-2}}, A(2) = (a_{i,j})_{\substack{i \neq n-1, n-2 \\ j \neq n, n-1}},$$

$$(2.2) \quad \det[A(k)] = 0 \text{ for all } k = 3, \dots, n-1$$

and

$$(2.3) \quad \begin{aligned} \det[A(k)] &= a_{n(n-k)} \det \left[(a_{i,j})_{\substack{i \neq n, n-1, n-2 \\ j \neq n, n-1, n-2}} \right] + \\ &\sum_{l=3}^{n-1} (-1)^l (a_{n-l, n-k}) \det \left[(a_{i,j})_{\substack{i \neq n-1, n-2, n-l \\ j \neq n, n-1, n-2}} \right] \end{aligned}$$

PROOF. Formula (2.1) is trivial by replacing in (1.2), k by 0, 1 and 2 respectively. Formula (2.2) is also trivial since the columns $(n-2)$ and $(n-k)$ are equal. Formula (2.3) is a simple developping of $\det A(k)$ according to the last column. \square

LEMMA 2. We have

$$(2.4) \quad B(0) = A,$$

$$(2.5) \quad \det[B(l)] = 0 \text{ for all } l = 1, \dots, n-1$$

and

$$(2.6) \quad \det[B(l)] = \sum_{k=0}^{n-1} (-1)^k (a_{n-l, n-k}) \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n-k}} \right].$$

PROOF. formula (2.4) is trivial by replacing in (1.2) l by 0. Formula (2.5) is also trivial since the rows n and $(n-1)$ are equal. Formula (2.6) is a simple developping of $\det B(l)$ according to the last row. \square

LEMMA 3. IF the determinant of the matrix $(a_{i,j})_{\substack{i \neq n-1,n \\ j \neq n-1,n}}$ is null, thus we get the following formula:

$$(2.7) \quad \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n-1}} \right] - \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n-1}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n}} \right] = 0$$

PROOF. To prove the formula (2.7) , we let $\det \left[(a_{i,j})_{\substack{i \neq n-1,n \\ j \neq n-1,n}} \right] = 0$. But $\det \left[(a_{i,j})_{\substack{i \neq n-1,n \\ j \neq n-1,n}} \right] = 0$ it means the rows of the matrix $(a_{i,j})_{\substack{i \neq n-1,n \\ j \neq n-1,n}}$ are dependent linearly. Consequently, $\exists (\lambda_i)_{i=1}^{n-2}$, $\exists k = 1, \dots, n-2$

$$(2.8) \quad \det \left[(a_{i,j})_{\substack{i \neq 2n-1-s \\ j \neq 2n-1-r}} \right] = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} & \dots & a_{1n-2} & a_{1r} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2k} & \dots & a_{2n-2} & a_{2r} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3k} & \dots & a_{3n-2} & a_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & \sum_{i=1}^{i=n-2} \lambda_i a_{ir} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \dots & a_{n-2,k} & \dots & a_{n-2,n-2} & a_{n-2,r} \\ a_{s,1} & a_{s,2} & a_{s,3} & \dots & a_{s,k} & \dots & a_{s,n-2} & a_{s,r} \end{vmatrix}$$

for $r, s = (n-1)$ or n . And also by calculating the determinant according to the last column, the formula will be the simple shape:

$$(2.9) \quad \begin{cases} \det \left[(a_{i,j})_{\substack{i \neq 2n-1-s \\ j \neq 2n-1-r}} \right] = \\ (-1)^{n-k+1} \left(\sum_{i=1}^{i=n-2} \lambda_i a_{ir} \right) \det \left[(a_{i,j})_{\substack{i \neq 2n-1-s,k \\ j \neq 2n-1-r,r}} \right] \end{cases} \quad \text{where } r, s = n-1 \text{ or } n$$

By using the formula (2.9) ,we get :

$$\begin{aligned} \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n}} \right] &= (-1)^{n-k+1} \left(\sum_{i=1}^{i=n-2} \lambda_i a_{in-1} \right) \det \left[(a_{i,j})_{\substack{i \neq n,k \\ j \neq n,n-1}} \right]; \\ \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n-1}} \right] &= (-1)^{n-k+1} \left(\sum_{i=1}^{i=n-2} \lambda_i a_{in} \right) \det \left[(a_{i,j})_{\substack{i \neq n-1,k \\ j \neq n-1,n}} \right]; \\ \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n-1}} \right] &= (-1)^{n-k+1} \left(\sum_{i=1}^{i=n-2} \lambda_i a_{in} \right) \det \left[(a_{i,j})_{\substack{i \neq n,k \\ j \neq n-1,n}} \right]; \\ \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n}} \right] &= (-1)^{n-k+1} \left(\sum_{i=1}^{i=n-2} \lambda_i a_{in-1} \right) \det \left[(a_{i,j})_{\substack{i \neq n-1,k \\ j \neq n,n-1}} \right]. \end{aligned}$$

Finally, the first member

$$\det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n-1}} \right] - \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n-1}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n}} \right]$$

of the formula (2.7) will be: $\left(\sum_{i=1}^{i=n-2} \lambda_i a_{in-1}\right) \left(\sum_{i=1}^{i=n-2} \lambda_i a_{in}\right) \cdot$
 $\left[\det \left[(a_{i,j})_{\substack{i \neq n,k \\ j \neq n,n-1}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1,k \\ j \neq n-1,n}} \right] - \det \left[(a_{i,j})_{\substack{i \neq n,k \\ j \neq n,n-1}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1,k \\ j \neq n-1,n}} \right] \right] =$
 0. And like this, we have finished the proof of the lemma 3. \square

THEOREM 1. *For all square matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ where $n > 2$ the formula (1.1) is satisfied*

PROOF. The formula (1.1) can be written as follows:

$$(2.10) \quad \det \left[(a_{i,j})_{1 \leq i,j \leq n} \right] \det \left[(a_{i,j})_{\substack{i \neq k,l \\ j \neq k,l}} \right] = \det \begin{bmatrix} \det \left[(a_{i,j})_{\substack{i \neq l \\ j \neq l}} \right] & \det \left[(a_{i,j})_{\substack{i \neq l \\ j \neq k}} \right] \\ \det \left[(a_{i,j})_{\substack{i \neq k \\ j \neq l}} \right] & \det \left[(a_{i,j})_{\substack{i \neq k \\ j \neq k}} \right] \end{bmatrix}.$$

To prove the formula (1.1), it is sufficient to prove the following formula:
 for $n > 2$ we have:

$$(2.11) \quad \det \left[(a_{i,j})_{1 \leq i,j \leq n} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1,n \\ j \neq n-1,n}} \right] = \\ \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n-1}} \right] - \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n-1}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n}} \right]$$

Because by pair replacing of row k with row $(n-1)$ and row l with row n , column k with column $(n-1)$ and column l with column n that does not change the determinant of the matrix $(a_{i,j})_{1 \leq i,j \leq n}$. And by applying the formula (2.11), then we replace some rows and some columns, noting that the number of the replaced rows is the same number of the replaced columns in order to stable the determinants, consequently, we get (1.1).

To prove the formula (2.11) there are two cases:

The first case: when $\det \left[(a_{i,j})_{\substack{i \neq n-1,n \\ j \neq n-1,n}} \right] = 0$, the proof of the formula (2.11) is the same proof of lemma3.

The second case: when $\det \left[(a_{i,j})_{\substack{i \neq n-1,n \\ j \neq n-1,n}} \right] \neq 0$ we prove the formula (2.11) inductively:

For $n = 3$, we find that the proof of the formula (2.11) is evident. For $n = 4$, we find that the proof of the formula (2.11) is evident with simple calculations.

When $n > 4$, we suppose the formula (2.11) is correct for $(n-1)$ and we prove it for n . In an other word, we prove that :

$$(2.12) \quad P - Q = \det \left[(a_{i,j})_{1 \leq i,j \leq n} \right] \cdot \det \left[(a_{i,j})_{\substack{i \neq n-1,n \\ j \neq n-1,n}} \right]$$

where

$$(2.13) \quad P = \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n-1}} \right]$$

and

$$(2.14) \quad Q = \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n-1}} \right] \det \left[(a_{i,j})_{\substack{i \neq n-1 \\ j \neq n}} \right].$$

We apply the formula (2.11) for $(n-1)$ on the formula (2.13)

$$(2.15) \quad \det \begin{bmatrix} (a_{i,j})_{i \neq n} \\ j \neq n \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1, n-2} \\ j \neq n, n-1, n-2 \end{bmatrix} = \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-1 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n, n-2 \end{bmatrix} - \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n, n-1 \end{bmatrix}.$$

$$(2.16) \quad \det \begin{bmatrix} (a_{i,j})_{i \neq n-1} \\ j \neq n-1 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1, n-2} \\ j \neq n, n-1, n-2 \end{bmatrix} = \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-1 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n-1, n-2} \\ j \neq n-1, n-2 \end{bmatrix} - \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n-1, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n-1, n-2} \\ j \neq n, n-1 \end{bmatrix}.$$

We apply the formula (2.11) for $(n-1)$ on the formula (2.14) :

$$(2.17) \quad \det \begin{bmatrix} (a_{i,j})_{i \neq n} \\ j \neq n-1 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1, n-2} \\ j \neq n, n-1, n-2 \end{bmatrix} = \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-1 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n-1, n-2 \end{bmatrix} - \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n-1, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n, n-1 \end{bmatrix}.$$

$$(2.18) \quad \det \begin{bmatrix} (a_{i,j})_{i \neq n-1} \\ j \neq n \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1, n-2} \\ j \neq n, n-1, n-2 \end{bmatrix} = \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-1 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n-1, n-2} \\ j \neq n, n-2 \end{bmatrix} - \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n-1, n-2} \\ j \neq n, n-1 \end{bmatrix}.$$

Using the formulas (2.15) -(2.18) on $(P-Q)$, we find :

$$(2.19) \quad (P-Q) \left(\det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1, n-2} \\ j \neq n, n-1, n-2 \end{bmatrix} \right)^2 = \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-1 \end{bmatrix} \cdot \left\{ \begin{aligned} &\Gamma \det \begin{bmatrix} (a_{i,j})_{i \neq n-1, n-2} \\ j \neq n-1, n-2 \end{bmatrix} - \Lambda \det \begin{bmatrix} (a_{i,j})_{i \neq n-1, n-2} \\ j \neq n, n-2 \end{bmatrix} \\ &+ F \det \begin{bmatrix} (a_{i,j})_{i \neq n-1, n-2} \\ j \neq n, n-1 \end{bmatrix} \end{aligned} \right\}$$

$$\text{where } \Gamma = \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-1 \end{bmatrix} - \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n, n-1 \end{bmatrix}$$

$$\Lambda = \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n-1, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-1 \end{bmatrix} - \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n-1, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n, n-1 \end{bmatrix}$$

$$F = \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n-1, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n, n-2 \end{bmatrix} - \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-2} \\ j \neq n, n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1} \\ j \neq n-1, n-2 \end{bmatrix}$$

We apply the formula (2.11) for $(n-1)$ on Γ, Λ and F , we find the following :

$$(2.20) \quad \Gamma = \det \begin{bmatrix} (a_{i,j})_{i \neq n} \\ j \neq n \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1, n-2} \\ j \neq n, n-1, n-2 \end{bmatrix}$$

$$(2.21) \quad \Lambda = \det \begin{bmatrix} (a_{i,j})_{i \neq n} \\ j \neq n-1 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1, n-2} \\ j \neq n, n-1, n-2 \end{bmatrix}$$

$$(2.22) \quad F = \det \begin{bmatrix} (a_{i,j})_{i \neq n} \\ j \neq n-2 \end{bmatrix} \det \begin{bmatrix} (a_{i,j})_{i \neq n, n-1, n-2} \\ j \neq n, n-1, n-2 \end{bmatrix}$$

In the formula (2.19) we replace Γ , Λ and F by their equivalents, we find:

$$(2.23) \quad (P - Q) \det \left[(a_{i,j})_{\substack{i \neq n, n-1, n-2 \\ j \neq n, n-1, n-2}} \right] = \det \left[(a_{i,j})_{\substack{i \neq n, n-1 \\ i \neq n, n-1}} \right] \cdot \left\{ \begin{array}{l} \det \left[(a_{i,j})_{\substack{i \neq n-1, n-2 \\ j \neq n-1, n-2}} \right] \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n}} \right] \\ - \det \left[(a_{i,j})_{\substack{i \neq n-1, n-2 \\ j \neq n, n-2}} \right] \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n-1}} \right] \\ + \det \left[(a_{i,j})_{\substack{i \neq n-1, n-2 \\ j \neq n, n-1}} \right] \det \left[(a_{i,j})_{\substack{i \neq n \\ j \neq n-2}} \right] \end{array} \right\}.$$

By using the formula (2.1) in lemma 1, we can write the formula (2.23) as follows:

$$(P - Q) \det \left[(a_{i,j})_{\substack{i \neq n, n-1, n-2 \\ j \neq n, n-1, n-2}} \right] = \det \left[(a_{i,j})_{\substack{i \neq n, n-1 \\ j \neq n, n-1}} \right] \sum_{k=0}^2 (-1)^k \det [A(k)] \det \left[(a_{i,j})_{\substack{i \neq n \\ i \neq n-k}} \right].$$

By using the formula (2.2) in lemma 1, we can write the above formula as follows:

$$(P - Q) \det \left[(a_{i,j})_{\substack{i \neq n, n-1, n-2 \\ j \neq n, n-1, n-2}} \right] = \det \left[(a_{i,j})_{\substack{i \neq n, n-1 \\ j \neq n, n-1}} \right] \sum_{k=0}^{n-1} (-1)^k \det [A(k)] \det \left[(a_{i,j})_{\substack{i \neq n \\ i \neq n-k}} \right].$$

By using the formula (2.3) in lemma 1 and the formula (2.6) in lemma 2, we can write the above formula as follows

$$(P - Q) \det \left[(a_{i,j})_{\substack{i \neq n, n-1, n-2 \\ j \neq n, n-1, n-2}} \right] = \det \left[(a_{i,j})_{\substack{i \neq n, n-1, n-2 \\ j \neq n, n-1, n-2}} \right] \det \left[(a_{i,j})_{\substack{i \neq n, n-1 \\ j \neq n, n-1}} \right] \det \left[(a_{i,j})_{1 \leq i, j \leq n} \right] \\ + \sum_{l=3}^{l=n-1} (-1)^l \{ \det B(l) \} \det \left[(a_{i,j})_{\substack{i \neq n-1, n-2, n-l \\ j \neq n, n-1, n-2}} \right].$$

And finally, by using the formula (2.5) in lemma 2, we find that the above formula will be formula (2.12). \square

References

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